

From Ballistic to Brownian Vortex Motion in Complex Oscillatory Media

Jörn Davidsen,¹ Ronaldo Erichsen,² Raymond Kapral,^{1,3} and Hugues Chaté⁴

¹Max-Planck-Institut für Physik Komplexer Systeme, Nöthnitzer Strasse 38, 01187 Dresden, Germany

²Universidade Estadual do Rio Grande do Sul, Porto Alegre, Brazil

³Chemical Physics Theory Group, Department of Chemistry, University of Toronto, Toronto, Ontario M5S 3H6, Canada

⁴CEA-Service de Physique de l'Etat Condensé, Centre d'Etudes de Saclay, 91191 Gif-sur-Yvette, France

(Received 17 February 2004; published 2 July 2004)

We show that the breaking of the rotation symmetry of spiral waves in two-dimensional complex (period-doubled or chaotic) oscillatory media by synchronization defect lines (SDLs) is accompanied by an intrinsic drift of the pattern. Single vortex motion changes from ballistic flights at a well-defined angle from the SDLs to Brownian-like diffusion when the turbulent character of the medium increases. It gives rise, in nonturbulent multispiral regimes, to a novel “vortex liquid.”

DOI: 10.1103/PhysRevLett.93.018305

PACS numbers: 82.40.Ck, 45.70.Qj, 89.75.Kd, 82.40.Bj

Chemical waves in reaction-diffusion systems with local excitable or simple oscillatory dynamics have been investigated extensively both theoretically and experimentally because of their relevance for a variety of physical, chemical, and biological processes [1,2]. In (quasi-)two-dimensional situations, spiral wave patterns are especially prevalent and important. They determine the characteristics of processes such as surface catalytic oxidation reactions [3], contraction of the heart muscle [4], and various signalling mechanisms in biological systems [5], to name only a few examples. The dynamics of the core or “vortex” of spiral waves plays an important role in many of these phenomena: some mechanisms for spiral breakup arise from core motion (see, for example, Ref. [6]) and moving spirals have been suggested to be responsible for some cardiac arrhythmias [7]. However, whereas the meandering instability of spiral cores in excitable media is well known [8], studies of vortex motion are still few in the oscillatory case, being limited to the unbounded acceleration characteristic of the core instability [9] or to erratic motion induced by spatio-temporal chaos or external noise [10,11].

When the local oscillations are not simple but possess a complex-periodic or chaotic character as observed, for example, in chemically reacting systems [12], spiral waves contain synchronization defect lines (SDLs) [13] separating domains of different oscillation phases and across which the phase changes by multiples of 2π . SDLs have been observed in experiments on Belousov-Zhabotinsky reaction [14,15]. While their origin and classification have been investigated, little is known about how they influence the spiral core to which they are connected.

In this Letter, we show that in complex oscillatory media the emergence of SDLs is accompanied by spiral core motion. The SDL breaks the rotational symmetry of the spiral wave, giving rise to a generic mechanism for core motion that differs from the instabilities that cause the nonsaturating core instability in simple oscillatory media and the meandering instability in simple excitable

media. We show that spirals in complex oscillatory media move ballistically in directions that are determined by the SDLs attached to the cores. In the regime of SDL-mediated turbulence where SDLs are spontaneously generated, the core motion is more complicated, leading to randomly oriented flights of random duration. With stronger turbulence this dynamics leads to vortex Brownian motion characterized by a well-defined diffusion constant. We finally show that multispiral configurations in the locally nonturbulent regime lead to a spatiotemporally chaotic “vortex liquid.”

Consider the reaction-diffusion system

$$\partial_t \mathbf{c}(\mathbf{r}, t) = \mathbf{R}[\mathbf{c}(\mathbf{r}, t)] + D \nabla^2 \mathbf{c}(\mathbf{r}, t),$$

where $\mathbf{c}(\mathbf{r}, t)$ is a vector of time-dependent concentrations at point \mathbf{r} in a two-dimensional domain of length/diameter L [16], D is the diffusion coefficient (taken to be the same for all species), and the local kinetics is specified by the vector-valued function $\mathbf{R}[\mathbf{c}(\mathbf{r}, t)]$. As a paradigmatic example of a system with complex local dynamics, we take $\mathbf{R}(\mathbf{c})$ to be given by the Rössler model [17] with $R_x = -c_y - c_z$, $R_y = c_x + 0.2c_y$, and $R_z = c_x c_z - C c_z + 0.2$. For $C \in [2.0, 6.0]$, the medium undergoes period-doubling bifurcations transforming the local dynamics from simple oscillatory to period-doubled orbits to chaotic dynamics. The rotational symmetry of spiral wave patterns is then broken by the ineluctable appearance of SDLs [13]: In a simple oscillatory medium, one turn of the spiral wave corresponds to one period of the local oscillation in phase space (change of 2π in the phase). When this spiral wave pattern undergoes a period-doubling bifurcation, the period of the local orbits doubles (change of 4π in the phase). Continuity in the medium then forces the emergence of a narrow strip connected to the core, the SDL, across which the phase jumps by 2π . Two oscillations are needed to return the medium to its original state. Along the SDLs, the local dynamics is period 1 (P1), while in the remainder of the medium it is period 2 (P2). An example of a P1 SDL in a

P2 medium is shown in the left panel of Fig. 1. In the period 4 (P4) regime, two types of SDLs are possible: lines where the phase jumps by 2π (local P1 dynamics) and lines where the phase jumps by 4π (local P2 dynamics). More generally, in a medium with period 2^n dynamics, 2^{n-1} SDLs with periodicities 2^k , ($k < n$), may exist, although for the Rössler medium only the P4 regime is observed before the local dynamics becomes chaotic. In these chaotic regimes the P1 and P2 SDLs persist. Deeper in the chaotic regime, turbulent states are found where SDLs are spontaneously created and annihilated [13].

First, we consider a single spiral in a disc-shaped domain with no-flux boundary conditions and focus on the complex oscillatory regime where no P1 SDLs are spontaneously generated ($3.03 < C < 4.557$). Our extensive simulations revealed that the simplest possible of the allowed configurations, where a single P1 SDL is attached to the core, is the unique asymptotic solution, even in the P4 and chaotic regimes where P2 SDLs exist. The presence of the SDL breaks the rotational symmetry and thus, on general grounds, one expects the spiral core to move [18]. This is indeed what is observed, albeit this motion is very slow, taking typically several thousand oscillations to move by one wavelength. After transients, the core moves ballistically at constant speed. In a finite system, it eventually encounters a boundary which influences its motion. Figure 2 shows that the speed v increases continuously and monotonically from the onset of P2 bulk oscillations, and analysis of the data indicates a power law behavior $v \sim (C - C_2)^\gamma$, where $C_2 \approx 3.03$ is the first period-doubling bifurcation and $\gamma = 1.5 \pm 0.02$. The angle α between the direction of motion and the attached SDL is constant (Fig. 1) and gradually decreases from 180° to 90° with increasing C , as shown in Fig. 2.

In the P4 and higher regimes both P2 and P1 SDLs may exist. We observed that both α and v are determined solely by the attached P1 SDL and are unaffected by

any P2 SDL that may be attached to the core, be it during a transient or in the regime where P2 SDLs are spontaneously and continuously generated ($4.306 \leq C < 4.557$). We conclude that, if it exists at all, the effect of an attached P2 SDL is much weaker than that of the dominant P1 line and essentially undetectable.

Next, we consider the turbulent regimes where P1 SDLs are spontaneously created ($4.557 < C \leq 6.0$). The continuous creation and annihilation of P1 SDLs and their dynamics strongly influences the motion of the core. More complicated configurations arise, which involve more than one P1 SDL attached to the core, and the motion is no longer characterized by a simple angle α . Complex connection and reconnection events between P1 SDLs, as well as between SDLs and the core, continuously occur. Close to the onset of this regime, the trajectory often changes its direction abruptly, even though long periods of (apparent) ballistic motion still persist. When a single P1 SDL is attached to the core, the motion is ballistic with $v \geq 0.0017$ and α close to 90° . Yet, most of the time, the spiral core has three P1 SDLs attached to it [Fig. 3(c)] and short-lived configurations with a higher odd number of SDLs exist as well. Even though a configuration with three SDLs is unstable for $C < 4.557$, it has a long lifetime and the combination of this persistence, the continuous creation of P1 SDLs and the interaction with other nearby SDLs [Fig. 3(c)], explains why this configuration dominates for $4.557 < C \leq 6.0$. Interestingly, the direction of motion can change without any apparent variation in the configuration of the three attached P1 SDLs. For instance, during the change of direction highlighted by the arrow in Fig. 3(b), the core configuration shown in Fig. 3(c) does not change significantly, and the three SDLs remain attached to the core. This indicates that part of the core motion is influenced by the dynamics

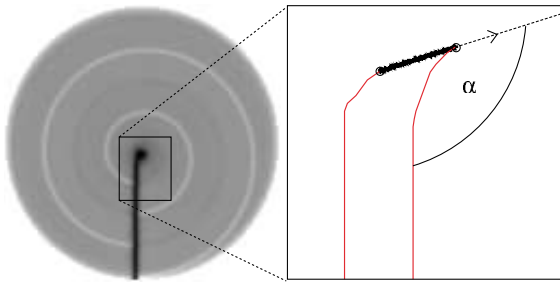


FIG. 1 (color online). Left: Snapshot of the scalar field $\Delta c_z(\mathbf{r}, t) = 1/\tau \int_0^\tau |c_z(\mathbf{r}, t+t') - c_z(\mathbf{r}, t+t'+\tau)| dt'$ in the P2 regime for $C = 3.5$, $L = 256$, and $D = 0.4$ (kept fixed throughout). The period of the P1 oscillation is $\tau = 5.95$. Black corresponds to $\Delta c_z \approx 0$ and indicates a SDL originating from the spiral core. Right: Magnification of the rectangular region in the left panel showing the SDL (dark grey or red lines) at two times. α is defined as the angle between the spiral core's trajectory (thick black line) and the attached SDL.

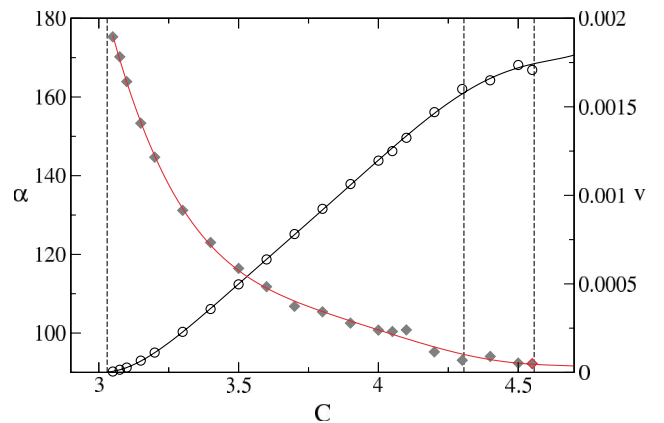


FIG. 2 (color online). Dependence of the angle α (solid diamonds) and the core velocity, v (circles) on the Rössler parameter C . The solid lines are guides to the eye. From left to right, the vertical lines mark the first period-doubling transition to the complex oscillatory regime and the transitions to the turbulent regimes (see text for details).

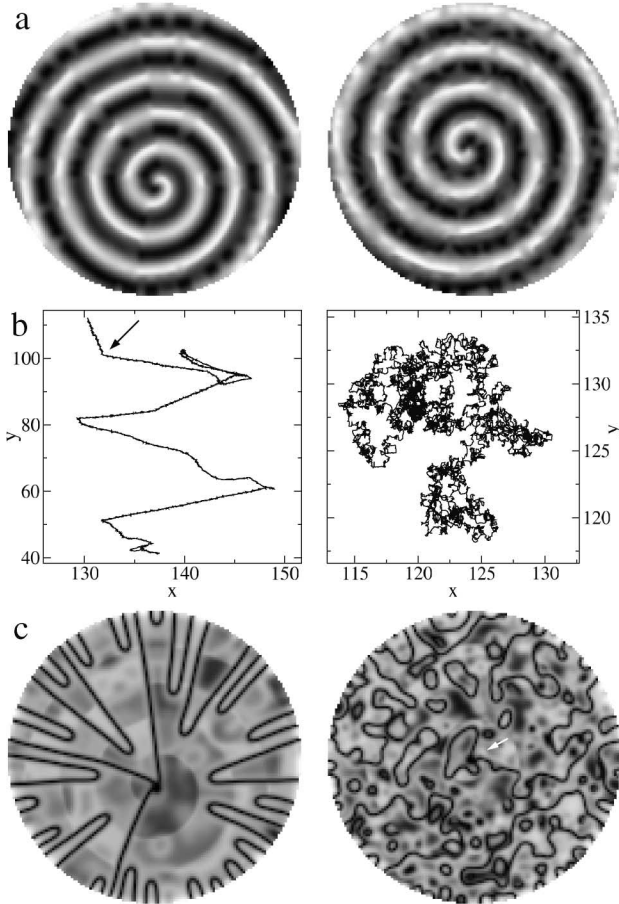


FIG. 3. Dynamics in the turbulent regime close to onset ($C = 4.6$, left) and far from onset ($C = 5.8$, right) for $L = 512$. (a) Snapshots of the c_x field. (b) Trajectories of the spiral core. The black arrow points to the position of the spiral core at the time when the snapshot of Δc_z was taken for $C = 4.6$ [shown in (c)]. (c) Snapshot of Δc_z visualizing the line defects. Note that in the configurations shown three P1 SDLs are attached to the core. The white arrow points to the position of the spiral core.

of the background field in the turbulent regime and/or that subtle rearrangements occur deep inside the core.

Deeper in the turbulent regime, the (apparent) ballistic flights become shorter and shorter and the core trajectory resembles Brownian motion [see Fig. 3(b)]. In this regime, SDLs are generated rapidly and homogeneously in the medium. This makes it possible to characterize the core motion by a well-defined diffusion constant D_v . Figure 4(a) shows the mean-squared displacement $\langle |\Delta \mathbf{r}_v(t)|^2 \rangle = \langle |\mathbf{r}_v(t) - \mathbf{r}_v(0)|^2 \rangle$ for different values of C . Here, $\langle \cdots \rangle$ denotes a time average over a long trajectory [19] and $\mathbf{r}_v(t)$ is the position of the core at time t . The dependence of $\langle |\Delta \mathbf{r}_v(t)|^2 \rangle$ on t is clearly linear, as expected for Brownian motion. The diffusion constant of the core motion determined from $\langle |\Delta \mathbf{r}_v(t)|^2 \rangle = 4D_v t$ decreases with increasing C [see Fig. 4(b)], reflecting the fact that changes of direction become more frequent while the mean velocity remains roughly constant.

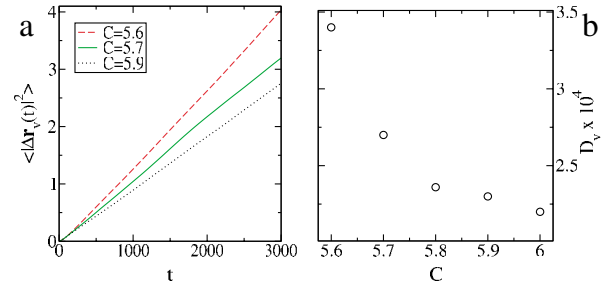


FIG. 4 (color online). (a) Mean squared displacement $\langle |\Delta \mathbf{r}_v(t)|^2 \rangle$ for different values of C in the turbulent regime. (b) The core diffusion coefficient D_v as a function of C .

We now turn to multispiral, spatially-disordered configurations which occur spontaneously in any large-enough domain. For oscillatory media described by the complex Ginzburg-Landau equation, it was shown recently that spatially-disordered frozen solutions do *not* exist [20]. Even when the single-spiral solution is stable, the weak but nontrivial effective interaction between spirals gives rise to ultraslow core motion. These results should apply to the Rössler medium in the simple oscillatory P1 regime ($C \in [2.0, 3.03]$). A long run performed at $C = 2.5$ [19] showed that after a transient period a residual and barely detectable core motion subsists, with typical velocities decreasing sharply with increasing spiral domain size indicating a rapidly decaying interaction. This observation, together with the wide variation of spiral domain sizes [Fig. 5(a)], suggests that the P1 Rössler medium is in the so-called “vortex glass” phase and not the “liquid” phase where all spiral domains have roughly the same size. Thus, core motion exists also in this regime without SDLs, albeit with velocities orders of magnitude smaller than those characteristic of core motion due to complex oscillations and can thus be neglected in the analysis of this latter case.

In a spatial domain with periodic boundary conditions, there must be an equal number of “positive” (clockwise) and “negative” (counterclockwise) spirals. Under P2 local dynamics, spirals have a tendency to be grouped in pairs (mostly of opposite sign) connected by a P1 SDL [Fig. 5(b)]. Inevitably, the pattern rearranges since a connected pair drifts apart, as shown in Fig. 5(c): a single-spiral pair in a square domain is connected by a straight P1 SDL. The two cores move with speed v at angles $\pm \alpha$ relative to the SDL and drift away from each other with speed $2v \cos(\pi - \alpha)$, which is positive as long as $|\alpha| > 90^\circ$. The sign of α is determined by the sign of the spiral. The dependence of v and α on C is indistinguishable from the single-spiral case (Fig. 2).

In multispiral disordered P2 regimes, our results show that spiral cores move nearly independently, stretching the P1 SDL to which they are attached, until they meet another core leading to reconnections and creation/annihilation of SDLs. Some “frustration” arises during this process, producing spiral cores connected by more

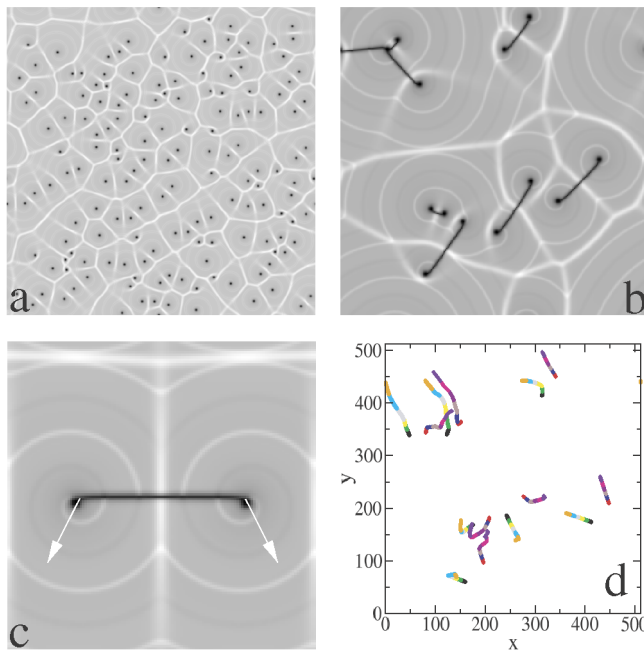


FIG. 5 (color online). (a) Snapshot of Δc_z showing a typical multispiral configuration in the P1 regime ($C = 2.5$, $L = 1024$). (b) Same as (a) but in the P2 regime, where P1 SDLs connect spiral cores ($C = 3.5$, $L = 512$). (c) Same as (b) but for a single pair of spirals connected by a P1 SDL ($L = 256$). The direction of the cores' motion is highlighted by the white arrows. (d) Trajectories of spiral cores leading to the configuration shown in (b). Every 3000 time units the color changes, starting with dark grey/red or black depending on the orientation of the spiral.

than one SDL. The rearrangements occur mainly when two cores are close and the cascade of reconnection events can span a distance of several wavelengths. In a P1 medium these cores would annihilate. On the contrary, here, because $\alpha > 90^\circ$, they repel each other after a new SDL connects them and the annihilation of spiral pairs is prevented [see Fig. 5(d)]. Thus, multispiral configurations in P2 media are characterized by a novel type of spatio-temporal chaos, which preserves the number of spiral cores through the complex dynamics of the P1 SDL connecting them.

Simulations show that this new vortex liquid state exists beyond the P2 regime, as long as α is significantly larger than 90° . For α close to 90° , annihilation of spiral pairs can occur, presumably due to the weaker repulsion effect induced by the P1 SDL. Crossing the transition point to the turbulent regime at $C = 4.557$, the annihilation rate of spiral pairs drastically increases. This is caused by the continuous generation of new P1 SDLs, which leads to permanent reconnections and effectively turns off the repulsion mechanism due to P1 SDLs.

The new types of spiral core motion and turbulence described here should be observable in distributed media with underlying complex dynamics. Studies of the more

realistic Willamowski-Rössler reaction scheme [21] support this statement. We also expect core motion to arise in excitable media supporting complex return to the rest state [22]. Experimentally, Belousov-Zhabotinsky reaction-diffusion systems are especially good candidates for studies of these phenomena since complex-periodic and chaotic regimes with SDLs have already been observed in the laboratory [15].

Research of R. K. supported in part by NSERC.

-
- [1] *Chemical Waves and Patterns*, edited by R. Kapral and K. Showalter (Kluwer Academic Publishers, Dordrecht, Netherlands, 1995).
 - [2] A. Mikhailov, *Foundations of Synergetics I: Distributed Active Systems* (Springer Verlag, New York, 1994); D. Walgraef, *Spatio-temporal Pattern Formation* (Springer Verlag, New York, 1997).
 - [3] R. Imbuhl and G. Ertl, *Chem. Rev.* **95**, 697 (1995).
 - [4] A.T. Winfree, *When Time Breaks Down* (Princeton University Press, Cambridge, NJ, 1987).
 - [5] A. Goldbeter, *Biological Oscillations and Cellular Rhythms* (Cambridge University Press, Cambridge, 1996).
 - [6] L. Brusch, A. Torcini, and M. Bär, *Phys. Rev. Lett.* **91**, 108302 (2003).
 - [7] R. A. Gray, J. Jalife, A.V. Panfilov, W.T. Baxter, C. Cabo, J.M. Davidenko, and A.M. Pertsov, *Science* **270**, 1222 (1995).
 - [8] D. Barkley, in Ref. [1], p. 163.
 - [9] I.S. Aranson and L. Kramer, *Rev. Mod. Phys.* **74**, 99 (2002).
 - [10] I.S. Aranson, H. Chaté, and L.-H. Tang, *Phys. Rev. Lett.* **80**, 2646 (1998).
 - [11] S. Alonso and F. Sagués, *Phys. Rev. E* **63**, 046205 (2001).
 - [12] S. Scott, *Chemical Chaos* (Oxford University Press, New York, 1991); J.-C. Roux and H.L. Swinney, in *Nonlinear Phenomena in Chemical Dynamics*, edited by C. Vidal and A. Pacault (Springer, New York, 1981); J.L. Hudson and J.C. Mankin, *J. Chem. Phys.* **74**, 6171 (1981).
 - [13] A. Goryachev, R. Kapral, and H. Chaté, *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **10**, 1537 (2000).
 - [14] M. Yoneyama, A. Fujii, and S. Maeda, *J. Am. Chem. Soc.* **117**, 8188 (1995).
 - [15] J.-S. Park and K. J. Lee, *Phys. Rev. Lett.* **83**, 5393 (1999); **88**, 224501 (2002).
 - [16] Our results are independent of L for $256 \leq L \leq 1024$ and of the discretization Δx for $0.5 \leq \Delta x \leq 1.0$.
 - [17] O. Rössler, *Phys. Lett. A* **57**, 397 (1976).
 - [18] B. Sandstede (private communication).
 - [19] Trajectory length was approximately 10^5 spiral periods.
 - [20] C. Brito, I.S. Aranson, and H. Chaté, *Phys. Rev. Lett.* **90**, 068301 (2003).
 - [21] K.D. Willamowski and O.E. Rössler, *Z. Naturforsch.* **35A**, 317 (1980).
 - [22] A. Goryachev and R. Kapral, *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **9**, 2243 (1999).