Math and Physics Review: Part 2

1 Calculus

1.1 Integration by parts

$$\int udv = uv - \int vdu$$

- Example:

$$\int_a^b dx\ f'(x)g(x) = f(x)g(x)\Big|_a^b - \int_a^b dx\ f(x)g'(x)$$

1.2 Change of Variable and Jacobians

Let $I = \int_{R_{xy}} dxdy\ \ f(x,y)$ be the integral over a connected region $R_{x,y}$. Change variables to u, v via the transform $g(u,v) = x$ and $h(u,v) = y$. It follows that:

$$I = \int_{R_{uv}} dudv\ f(g(u,v), h(u,v))\left|\frac{\partial(g,h)}{\partial(u,v)}\right|.$$

where the *Jacobian* $\frac{\partial(g,h)}{\partial(u,v)}$ is

$$\frac{\partial(g,h)}{\partial(u,v)} \equiv \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial u} \end{vmatrix}$$

- Example: Suppose

  $$I = \int_{-\infty}^{\infty} dxdy\ f(x)g(x-y).$$

  Let $u = x$ and $v = x - y$. Under this transformation, range of v is $(-\infty, \infty)$ at a fixed value of u (or x). The Jacobian J is

  $$J = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$$

Thus,

$$I = \int_{-\infty}^{\infty} dudv\ f(u)g(v)|-1| = \int_{-\infty}^{\infty} duf(u)\ \int_{-\infty}^{\infty} dvg(v).$$

1.3 Relative extrema

Suppose we have a continuous function f(x,y) and $f_x = \frac{\partial f}{\partial x} = 0$ and $f_y = 0$ at a point (a, b).

- If the discriminant

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2 > 0 \text{ at } (a, b)$$

  - $f_{xx} < 0$ at (a, b) implies a *relative maximum* at (a, b).
  - $f_{xx} > 0$ at (a, b) implies a *relative minimum* at (a, b).

- If $f_{xx} f_{yy} - f_{xy}^2 < 0$ then (a, b) is a *saddle point.*

1.4 Method of Langrange Multipliers

Maximize function $f(x_1, \ldots, x_n)$ subject to the constraint $g(x_1, \ldots, x_n) = 0$

- If no constraints, maximum satisfies

$$\frac{\partial f}{\partial x_i} = 0 \text{ for } i = 1, \ldots, n.$$

  Since $g(x_1, \ldots, x_n) = 0$, we also have

$$\frac{\partial g}{\partial x_i} = 0 \text{ for } i = 1, \ldots, n.$$

- It then follows that:

$$\frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0 \text{ for all } \lambda \text{ and for } i = 1, \ldots, n.$$

- Equation above plus the condition $g(x_1, \ldots, x_n) = 0$ gives $n + 1$ equations for $n + 1$ variables $(x_1, \ldots, x_n, \lambda)$.

- λ is called a *Lagrange multiplier.*

1.5 Functional derivatives

Consider the *functional* $F[G(x)] = \int dx\ g(x)^2$.

- Value depends on the functional form of $g(x)$.
- How does this value change if the function $g(x)$ is changed? Define $g(x,\alpha) = g(x) + \alpha\eta(x) = g(x) + \delta g(x)$.
- Change in value of functional due to change in functional form of $g(x)$ is

$$\begin{aligned}
\delta F \equiv F[g(x,\alpha)] - F[g(x)] &= \int dx\ \left[(g(x) + \alpha\eta(x))^2 - g(x)^2\right] \\
&= \int dx\ \left(2g(x)\overbrace{\alpha\eta(x)}^{\delta g(x)} + (\alpha\eta(x))^2\right) \\
&= \int dx\ \left(\frac{\partial f}{\partial g}\delta g(x) + \frac{1}{2}\frac{\partial^2 f}{\partial g^2}\delta g(x)^2\right),
\end{aligned}$$

  where $f = g^2$.

- What about the general case for $F[g(x)] = \int dx\ f(g(x))$?

$$\delta F = \int dx\ \left(\frac{\partial f}{\partial g}\delta g(x) + \frac{1}{2}\frac{\partial^2 f}{\partial g^2}\delta g(x)^2 + \frac{1}{3!}\frac{\partial^3 f}{\partial g^3}\delta g(x)^3 + \cdots\right)$$