Math Review

1 Taylor expansion

- Expand function $f(x+a)$ from small a around $a=0$.

$$\begin{aligned} f(x+a) &= f(x) + f'(x)a + \frac{1}{2}f''(x)a^2 + \cdots \\ &= \sum_{j=0}^{\infty} \frac{a^j}{j!} \frac{d^j}{dx^j} f(x+a)\Big|_{a=0}. \end{aligned}$$

- Since $e^{\lambda x} = \exp(\lambda x) = \sum_{j=0}^{\infty} x^j \lambda^j / j!$

$$f(x+a) = \exp\left(a\frac{d}{dx}\right) f(x). \tag{1}$$

2 Series expansions

For $|x| < 1$,

$$\begin{aligned} \frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots \\ \frac{1}{1-x} &= 1 + x + x^2 + \cdots \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\ \ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots \end{aligned}$$

3 Probability theory:

3.1 Discrete systems

Suppose have measurable E with n discrete values $E_1, E_2, \ldots, E_n$. Let

$$\begin{aligned} N &= \text{number of measurements} \\ N_i &= \text{number of measurements of } E_i. \end{aligned}$$

Then

$$P_i = \text{Probability that } E_i \text{ is measured} = \lim_{N\to\infty} \frac{N_i}{N} \equiv P(E_i)$$

Properties:

1. $0 \le P_i \le 1$
2. $\sum_{i=1}^{n} P_i = 1$

Averages:

$$\begin{aligned} \overline{E} &= \sum_{i=1}^{n} E_i P_i \\ \overline{E^2} &= \sum_{i=1}^{n} E_i^2 P_i \\ \overline{H(E)} &= \sum_{i=1}^{n} H(E_i) P_i \end{aligned}$$

Variance of E:

$$\begin{aligned} \sigma_E^2 &\equiv \overline{E^2} - \left(\overline{E}\right)^2 \\ &= \overline{\left(E_i - \overline{E}\right)^2} \end{aligned}$$

- σ_E^2 measures the dispersion of the probability distribution: how spread out values are.
- In general, $\sigma_E^2 \neq 0$ unless $P_i = \delta_{ij}$ for some j. This notation means:

$$P_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad \text{which implies } \overline{E} = E_i. \tag{2}$$

- Tchebycheff Inequality:

$$Prob\left(\left|E - \overline{E}\right| \ge \lambda \overline{E}\right) \le \frac{\sigma_E^2}{\lambda^2 \overline{E}^2}. \tag{3}$$

- Joint probability: Suppose N measurements of two properties E and G.

$$\begin{aligned} n_{ij} &= \text{number of measurements of } E_i \text{ and } G_j \\ P_{ij} &= \lim_{N\to\infty} \frac{n_{ij}}{N} \equiv P(E_i, G_j) \equiv \text{joint probability.} \end{aligned}$$

  Properties:

  1. $\sum_{i.j} P(E_i, G_j) = 1$.
  2. $\sum_i P(E_i, G_j) = P(G_j)$.
  3. $\sum_j P(E_i, G_j) = P(E_i)$.
  4. If E_i and G_j are independent, then $P(E_i, G_j) = P(E_i)P(G_j)$.

3.2 Combinatorics

- Fact 1: The number of permutations of N distinguishable objects is $N!$
- Fact 2: The number of ways of assigning N distinct objects into r distinct containers is

$$t = \frac{N!}{\prod_{i=1}^{r} N_i!} \tag{4}$$

  where N_i is the number of objects in the ith container.

  – Example: Number of ways of selecting k distinct objects from a larger set of n distinct objects is:

$$\begin{pmatrix} n \\ k \end{pmatrix} \equiv \frac{n!}{k!(n-k)!}$$

- Coin Tossing: Let

$$\begin{aligned} n &= \text{Number of tosses} \\ k &= \text{Number of heads} \end{aligned}$$

  then

$$\begin{aligned} P(k, n) &= \text{Probability of } k \text{ heads in } n \text{ tosses.} \\ &= \left(\frac{1}{2}\right)^n \begin{pmatrix} n \\ k \end{pmatrix} \end{aligned}$$

– Suppose the probability of winning is p and q is the probability of losing. What is the probability of winning k times in n games? Determined by "Bernoulli" or "binomial" probability.

$$(p+q)^n = p^n + p^{n-1}q \begin{pmatrix} n \\ n-1 \end{pmatrix} + p^{n-2}q^2 \begin{pmatrix} n \\ n-2 \end{pmatrix} + \cdots + q^n \quad (5)$$

Results:

$$P(k,n) = p^k q^{n-k} \begin{pmatrix} n \\ k \end{pmatrix} \quad (6)$$

$$\begin{aligned} \overline{k} &= np \\ \overline{k^2} &= np + n(n-1)p^2 \\ \sigma_k^2 &= npq \\ Prob\left(\left|k - \overline{k}\right| \geq \lambda \overline{k}\right) &\leq \frac{q}{np\lambda^2}. \end{aligned}$$

– Note that distribution narrows with n. Typical behavior if $\overline{k} \sim n$.

- Generating Functions We define the *generating function* of a distribution $P(k,n)$ to be

$$F(x) = \sum_{k=0}^{n} P(k,n) x^k. \quad (7)$$

Note that $F(1) = 1$ since distribution is normalized. If

$$P(k,n) = p^k q^{n-k} \begin{pmatrix} n \\ k \end{pmatrix} \qquad \text{then} \qquad F(x) = (q+px)^n. \quad (8)$$

Useful for calcuating *moments* of a distribution:

$$\begin{aligned} \overline{k} &= \left(xF'(x)\right)_{x=1} \\ \overline{k^l} &= \left[\overbrace{\left(x\frac{d}{dx}\right) \cdots \left(x\frac{d}{dx}\right)}^{l \text{ times}} F(x) \right]_{x=1} \end{aligned}$$

3.3 Continuous Systems

- Probability of measure an observable X with values between $x, x + dx$ is $p(x)dx$. $p(x)$ is called the "probability density".
  Properties:
  1. Positive definite: $p(x) \geq 0$.
  2. Normalized: $\int_{-\infty}^{\infty} dx p(x) = 1$
- Averages:

$$\begin{aligned} \overline{x} &= \int_{-\infty}^{\infty} dx\, x p(x) \qquad \overline{f(x)} = \int_{-\infty}^{\infty} dx\, f(x)p(x) \\ \sigma_x^2 &= \overline{x^2} - \overline{x}^2 = \int_{-\infty}^{\infty} dx \left(x^2 - \overline{x}^2\right) p(x) \end{aligned}$$

- Example probability density: $p(x) = ce^{-\alpha x^2}$
- Properties:

$$c = \sqrt{\frac{\alpha}{\pi}}$$

$$\sigma_x^2 = \frac{1}{2\alpha}$$

  so

$$p(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{x^2}{2\sigma_x^2}}.$$

  - What happens when $\sigma_x^2 \to 0$? Infinitely narrow distribution, called a *dirac delta function*. Probability density has all the weight on one value.
  - There are other representations of the dirac delta function: basically defined in such a way that *one* value receives all the weight.
- Delta functions: defined in a limiting sense.

$$\delta^{(\epsilon)}(x) = \begin{cases} \frac{1}{\epsilon} & -\frac{\epsilon}{2} \leq x \leq \frac{\epsilon}{2} \\ 0 & |x| > \frac{\epsilon}{2} \end{cases} \qquad \int_{-\infty}^{\infty} dx\, \delta^{(\epsilon)}(x) = \int_{-\epsilon/2}^{\epsilon/2} dx \frac{1}{\epsilon} = 1.$$

$$\int_{-\infty}^{\infty} dx\, \delta^{(\epsilon)}(x) f(x) \approx f(0) \int_{-\infty}^{\infty} dx\, \delta^{(\epsilon)}(x) = f(0) \qquad \text{if } \epsilon \ll 1.$$

- Function $f(x)$ essentially constant over infinitesimal interval.
- Definition of delta function: $\delta(x) = \lim_{\epsilon \to 0} \delta^{(\epsilon)}(x)$.

• Representations of delta function in limit $\epsilon \to 0$:

1. $\frac{1}{2\epsilon} e^{-|x|/\epsilon}$
2. $\frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2}$
3. $\frac{1}{\epsilon\sqrt{\pi}} e^{-x^2/\epsilon^2}$
4. $\frac{1}{\pi} \frac{\sin x/\epsilon}{x}$

- For any continuous function f of x, for all forms above we get

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dx\ \delta^{(\epsilon)}(x - x_0) f(x) = f(x_0).$$

Some properties of the delta function

1. $\delta(-x) = \delta(x)$
2. $\delta(cx) = \frac{1}{|c|}\delta(x)$
3. $\delta[g(x)] = \sum_j \frac{\delta(x-x_j)}{|g'(x_j)|}$ where $g(x_j) = 0$ and $g'(x_j) \neq 0$.
4. $g(x)\delta(x - x_0) = g(x_0)\delta(x - x_0)$
5. $\int_{-\infty}^{\infty} dx\ \delta(x - y)\delta(x - z) = \delta(y - z)$
6. $\int_{-\infty}^{\infty} dx\ \frac{d\delta(x-x_0)}{dx} f(x) = -\int_{-\infty}^{\infty} dx\ \delta(x - x_0) f'(x) = -f'(x_0)$